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MODULAR-FUNCTION REPRESENTATION OF THE SOLUTIONS OF THE PROBLEM OF TWO FIXED CENTRES†

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The problem of two fixed centres can be integrated in quadratures by using Stoeckel's theorem in spheroidal coordinates. The elliptic coordinates of the meridian plane satisfy equations that define a complex elliptic curve. The solution can be written down by using the uniformization formula of the elliptic curve. The constants defining the result in terms of doubly periodic functions are calculated successively from the initial data. The half-period ratio is determined by solving a complex transcendental equation using modular functions. The magnitude of the half-period of least absolute value is determined using ϑ-functions of the zero argument, using invariants of the Weierstrass $\wp$-function.

1. REDUCTION OF THE PROBLEM

LET THE motion of a material point be defined in (generally complex) three-dimensional space in terms of the force function

$$U = m_1/r_1 + m_2/r_2 \tag{1.1}$$

where m_1 and m_2 are the "masses" of the centres of gravitation, and r_1 and r_2 are the distances from the point to the centres, which lie on the z-axis at points with coordinates $(0, 0, c_1)$ and $(0, 0, c_2)$, respectively. It is assumed that

$$r_k = x_1^2 + x_2^2 + (x_3 - c_k)^2)^{1/2} \quad (k = 1, 2)$$

We shall also assume from the start that, by a suitable choice of the units of measurement, the gravitational constant is unity.

We know [1] that (1.1) is a real function, i.e. it may describe real motions, in only two cases: the classical case (when the masses and coordinates of the centres are real) and a generalized case (the masses and coordinates are complex conjugate).

We will continue our analysis in a uniform way for either of the two admissible cases. To fix our ideas, therefore, we will consider the generalized problem of two fixed centres, also assuming throughout that

$$m_1 = \frac{m}{2}(1 + i\sigma), \quad m_2 = \frac{m}{2}(1 - i\sigma), \quad c_1 = c(\sigma + i), \quad c_2 = c(\sigma - i)$$

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By Stoeckel's theorem, the problem can be integrated in quadratures in oblate spheroidal coordinates λ, μ, ν, defined by the formulae

$$x_1 = c[(1+\lambda^2)(1-\mu^2)]^{1/2}\cos\nu$$
$$x_2 = c[(1+\lambda^2)(1-\mu^2)]^{1/2}\sin\nu$$
$$x_3 = c\sigma + c\lambda\mu; \quad \lambda \geqslant 0, \quad -1 \leqslant \mu \leqslant 1, \quad 0 \leqslant \nu < 2\pi$$

In terms of these coordinates the kinetic energy and force function may be written in the form

$$K = \frac{c^2}{2}\left[\frac{\lambda^2+\mu^2}{1+\lambda^2}\dot{\lambda}^2 + \frac{\lambda^2+\mu^2}{1-\mu^2}\dot{\mu}^2 + (1+\lambda^2)(1-\mu^2)\dot{\nu}^2\right]$$
$$U = \frac{m}{c}\frac{\lambda-\sigma\mu}{\lambda^2+\mu^2}$$

An application of Stoeckel's theorem, followed by the introduction of a new independent variable u related to the time t by the equation

$$dt/du = c(\lambda^2+\mu^2) \tag{1.2}$$

shows that the elliptic coordinates of the meridian plane λ, μ will satisfy the differential equations

$$\frac{1}{2}\left[\frac{d\lambda}{du}\right]^2 = \left[\frac{m}{c}\lambda + \alpha_1\lambda^2 + \alpha_2\right](1+\lambda^2) + \alpha_3$$
$$\frac{1}{2}\left[\frac{d\mu}{du}\right]^2 = \left[-\frac{m\sigma}{c}\mu + \alpha_1\mu^2 + \alpha_2\right](1+\mu^2) - \alpha_3 \tag{1.3}$$

where α_1, α_2, and α_3 are constants of integration that define an invariant toroidal manifold in the phase space of the problem: α_1 is the value of the energy integral, and α_3 corresponds to the area constant and determines the rotation of the meridian plane via the equation

$$\left[\frac{d\nu}{du}\right]^2 = 2\alpha_3\left[\frac{1}{1-\mu^2} - \frac{1}{1+\lambda^2}\right] \tag{1.4}$$

The quantity c characterizes the distance between the centres of gravitation. In space-satellite problems c must be a small quantity, while the dimensions of the orbit are of the order of unity.

2. ELLIPTIC CURVES

Either of Eqs (1.3) may be written in the form

$$\left[\frac{dz}{du}\right]^2 = [\alpha z + \alpha_1 z^2 + \alpha_2](\delta + z^2) + \alpha_3 \tag{2.1}$$

where the coefficients on the right are related in a fairly obvious way to the coefficients of (1.3).

Equation (2.1) describes an elliptic curve in the complex space $\mathbf{C}^2$ of the variables (z, $w = dz/du$). A general solution of (2.1) may be obtained by using a uniformization procedure [2]. To that end we have to submit (2.1) to a series of transformations. We first bring (2.1) to the general form

$$\left[\frac{dz}{du}\right]^2 = a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4$$
$$a_0 = \alpha_1,\quad a_1 = \frac{\alpha}{4},\quad a_2 = \frac{\alpha_1\delta + \alpha_2}{6},\quad a_3 = \frac{\alpha\delta}{4},\quad a_4 = \alpha_2\delta + \alpha_3 \tag{2.2}$$

changing to a new independent variable

$$u_1 = a_0^{1/2} u \tag{2.3}$$

we convert (2.2) to the equation

$$\left[\frac{dz}{du_1}\right]^2 = z^4 + 4a_1' z^3 + 6a_2' z^2 + 4a_3' z + a_4' \tag{2.4}$$

$$a_1' = \frac{\alpha}{4\alpha_1},\quad a_2' = \frac{\alpha_1\delta + \alpha_2}{6\alpha_1},\quad a_3' = \frac{\alpha\delta}{4\alpha_1},\quad a_4' = \frac{\alpha_2\delta + \alpha_3}{\alpha_1} \tag{2.5}$$

Making the substitution $z \to z_1$, where

$$z_1 = z + a_1' \tag{2.6}$$

we eliminate the third-degree term in (2.4), obtaining

$$\left[\frac{dz_1}{du_1}\right]^2 = z_1^4 - 6Az_1^2 + 4Bz_1 + C \tag{2.7}$$

$$A = a_1'^2 - a_2',\quad B = 2a_1'^3 - 3a_1'a_2' + a_3',$$
$$C = -3a_1'^4 + 6a_1'^2 a_2' - 4a_1'a_3' + a_4' \tag{2.8}$$

Now, proceeding as in [2], we introduce the quantities

$$g_2 = C + 3A^2,\quad g_3 = -AC + A^3 - B^2 \tag{2.9}$$

which will be treated as invariants of the Weierstrass $\wp$-function; since $B^2 = 4A^3 - g_2 A - g_3$, the system of equations

$$\wp(\upsilon) = A,\qquad \wp'(\upsilon) = B \tag{2.10}$$

has a unique solution υ in the parallelogram of periods.

Further, using the addition theorem for $\wp$-functions we obtain a uniformization of the curve (2.7), in the form

$$z_1(u_1) = \frac{1}{2}\frac{\wp'(u_1 - \upsilon/2) - \wp'(\upsilon)}{\wp(u_1 - \upsilon/2) - \wp(\upsilon)}$$

$$\frac{dz_1}{du_1}(u_1) = \wp(u_1 - \upsilon/2) - \wp(u_1 + \upsilon/2) \tag{2.11}$$

Thus, suppose we have obtained a solution of (2.11), which in fact amounts to calculating the half-periods of the $\wp$-function. We now resume our task of finding a solution of the original problem, given the following initial data when $t = t_0$: x_{10}, x_{20}, x_{30}, $\dot{x}_{10}$, $\dot{x}_{20}$, $\dot{x}_{30}$. The initial data in spheroidal coordinates are readily calculated. Using (1.2), we determine the initial data for Eqs (1.3). Since they are of the same type, we confine our attention to Eq. (2.1).

Suppose we have calculated the values of z_0 and $(dz/du)_0 = z_0'$. These two numbers determine a point on the elliptic curve defined parametrically as follows:

$$z(u) = \frac{1}{2\beta}\frac{\wp'(u-w/2)-\wp'(w)}{\wp(u-w/2)-\wp(w)}$$
$$\frac{dz}{du}(u) = \beta^{-1}[\wp(u-w/2)-\wp(u+w/2)] \tag{2.12}$$
$$\beta = \alpha_1^{1/2}, \quad w = \mathfrak{v}/\beta$$

The initial values for this curve correspond to a value of the parameter $u = \upsilon_0$ that satisfies the system of equations

$$z(\upsilon_0) = z_0, \quad z'(\upsilon_0) = z_0' \tag{2.13}$$

Using (2.12), we can now express the solution of the Cauchy problem in the form

$$z = z(\upsilon_0 + u) \tag{2.14}$$

Thus, the parameter value corresponding to the starting time $t = t_0$ is $u = 0$.

The systems of equations (2.13) and (2.10) can be solved numerically. This can be done using the parity properties of the function (2.13) and the $\wp$-function, respectively.

It is obvious from (2.3) that the procedure does not work when $\alpha_1 = 0$ (the case of zero energy). But even then, on the assumption that $a_1 \neq 0$, it is fairly easy to solve Eq. (2.1). We define a new independent variable u_1 by

$$u_1 = a_1^{1/2} u \tag{2.15}$$

Then Eq. (2.4) becomes

$$\left[\frac{dz}{du_1}\right]^2 = 4z^3 + 6a_2'z^2 + 4a_3'z + a_4' \tag{2.16}$$

$$a_2' = \frac{2\alpha_2}{3\alpha}, \quad a_3' = \delta, \quad a_4' = 4\frac{\alpha_2\delta + \alpha_3}{\alpha} \tag{2.17}$$

To reduce Eq. (2.16) to Weierstrass form, we must eliminate the quadratic term. This can be done by a linear transformation

$$z_1 = z + a_2'/2 \tag{2.18}$$

After this substitution the equation of the elliptic curve will be in Weierstrass normal form

$$\left[\frac{dz_1}{du_1}\right]^2 = 4z_1^3 - g_2 z_1 - g_3 \tag{2.19}$$

$$g_2 = 3a_2'^2 - 4a_3', \quad g_3 = -a_2'^3 + 4a_2' a_3' - a_4' \tag{2.20}$$

The curve (2.19) is uniformized by the equations

$$z_1(u_1) = \wp(u_1), \quad \frac{dz_1}{du_1}(u_1) = \wp'(u_1) \tag{2.21}$$

Cases of further degeneracy of the curve (2.2) are trivial.

3. MODULAR FUNCTIONS

The $\wp$-function in Eqs (2.12), (2.21) depends on structural parameters: the half-periods ω and ω'. We will assume that $\mathrm{Im}(\omega'/\omega) > 0$, while $|\omega| \leq |\omega'|$. This can always be achieved by a suitable choice on the lattice of periods in **C**. Denote the values of the invariants g_2, g_3 obtained by formulae (2.9) or (2.20) by a and b, respectively.

If the half-periods are known, the invariants may be represented by Eisenstein series

$$g_2(\omega,\omega') = 60\Sigma'(m\omega + m'\omega')^{-4}$$
$$g_3(\omega,\omega') = 140\Sigma'(m\omega + m'\omega')^{-6} \tag{3.1}$$

where the summation is carried out over m, $m' \in \mathbf{Z}$, and the prime on the summation symbol indicates that the term with both indices zero $(m, m') = (0, 0)$ must be omitted.

From the computational point of view, it is more convenient to use as parameters not ω and ω' but τ, and ω, where $\tau = \omega'/\omega$. By construction, $\tau \in H$, where H is the upper half-plane. On the upper half-plane we can define a modular function $J(\tau)$ in terms of the functions (3.1)

$$J(\tau) \rightleftharpoons \frac{g_2^3(1,\tau)}{g_2^3(1,\tau) - 27g_3^2(1,\tau)} \tag{3.2}$$

We know [3] that $J(\tau)$ is invariant under the modular group Γ. The action of Γ on the upper half-plane H divides it into domains, each consisting of points that are congruent modulo Γ (shown hatched in Fig. 1).

We also know that Γ is generated by (besides the identity) two transformations

$$T: \tau \to \tau + 1, \quad S: \tau \to -1/\tau, \tag{3.3}$$

The fundamental domain G is bounded by the unit circle and by two straight lines $\mathrm{Re}\,\tau = \pm\frac{1}{2}$. In addition, the following part of the boundary belongs to the domain (see Fig. 1)

$$AC = \{\tau\colon \mathrm{Re}\,\tau = -\tfrac{1}{2},\ |\tau| > 1\}, \qquad A = \{\tau = -\tfrac{1}{2} + i3^{1/2}/2\}$$
$$AB = \{\tau\colon |\tau| = 1,\ -\tfrac{1}{2} < \mathrm{Re}\,\tau < 0\}, \quad B = \{\tau = i\}$$

The remainder of the boundary may be obtained from the above by the transformations T, S.

It now remains to recall the well-known theorem [1, 3] according to which the equation

$$J(\tau) = c \tag{3.4}$$

has exactly one solution in G. The cases $c = 0$, 1 and ∞ require special consideration.

If $J(\tau) = 0$, then $\tau = e^{2\pi i/3}$, while if $J(\tau) = 1$, we have $\tau = i$; these values of τ are triple and double roots, respectively, of Eq. (3.4). Since A is common to three domains (of the six for which it is a limit point)—the images of G under the elements of Γ, while B is common to two, we may legitimately state that the function $J\colon G \to \mathbf{C}$ is both surjective and injective, i.e. it is a bijection.

We will now describe an algorithm that, given the coefficients of (2.2), computes the parameter τ. First, we find the values a and b of the invariants g_1 and g_2 (see the previous section). We then evaluate $J(\tau)$ from the expression $a^3/(a^3 - 27b^2)$. If it turns out that $J(\tau) = 0$, 1, ∞, the computation of τ is complete and we accordingly put $\tau = e^{2\pi i/3}$, i, ∞. Of course, in actual practice exact equality is impossible, so we must take computer errors into account.

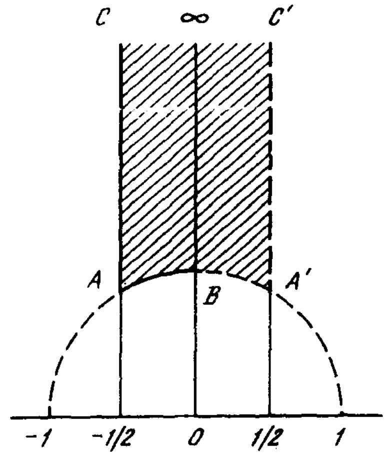

FIG. 1.

By (3.2), it is clear that $J(\tau)=0$, if $a=0$, $b\neq 0$; $J(\tau)=1$ if $b=0$, $a\neq 0$; $J(\tau)=\infty$ if $a\neq 0$, $a^3-27b^2=0$. The limiting case $a=b=0$ requires further analysis. By (3.2), we can write

$$J(\tau)=(1-27\alpha)^{-1} \quad (\alpha=b^2/a^3)$$

If the parameters of the problem are varied so that $\alpha\to 1/27$, then $J(\tau)\to\infty$. As $\alpha\to 0$, ∞ we obtain $J(\tau)\to 1$, 0, respectively, that is, the situation is reduced to that already described.

Our problem is thus to search for the unique root in G of Eq. (3.4), where the cases $c=0$, 1 and ∞ have already been disposed of. To that end, we can use any standard method for the numerical solution of transcendental equations, such as Newton's method. The initial approximation may be a point of the interior of G, e.g. $\tau_0=2i\in G$. Further iterations will then "close in" on the nearest root of Eq. (3.4).

It is not impossible that this root may turn out to lie not in G but in another domain—the image of G under the action of Γ. This may happen if, owing to the specific method chosen, the initial approximation τ_0 is actually closer to a root of (3.4) outside G.

Note that, in any case, the iterations will not leave the upper half-plane, since G is separated from the real axis by other domains in which the algorithm will inevitably approximate a root (provided, of course, that the step size is sufficiently small).

In fact, the elements of Γ can map the point ∞ of G either onto ∞ (transformation T) or onto O (transformation S). By combining T and S in all possible ways, therefore, we obtain all images of ∞ only at rational points of the real axis. Since the images of G become increasingly finer as one approaches the axis $\mathrm{Im}\,\tau=0$, while the function $J(\tau)$ is a congruence in each of them, $J:G\to\mathbf{C}$, it follows that this function behaves in an increasingly irregular fashion as $\mathrm{Im}\,\tau\to 0$. As one approaches the real axis within a single domain, $J(\tau)\to\infty$. Thus, an accurately organized computational procedure will "sense" this barrier of irregularity as $\mathrm{Im}\,\tau\to 0$. In the final analysis, one can always check the value of Im τ and, at a sufficiently small distance from the axis $\mathrm{Im}\,\tau=0$, modify the initial approximation.

Thus, let $\tau^*\in H$ be the numerically computed root of Eq. (3.4). For our further calculations we will need the point $\tau\in G$ congruent to τ^* relative to the group Γ. To solve that problem, we will use the transformations T and S. The algorithm consists of the following steps.

1. If $\tau^*\in G$, set $\tau=\tau^*$. End.
2. If τ^* lies on the arc $[A'B)$, apply the transformation S: $\tau=-1/\tau^*$. End.
3. If τ^* lies on the arc $(A'C')$, apply the transformation $T^{-1}:\tau=\tau^*-1$. End.
4. If $\mathrm{Re}\,\tau^*<-\tfrac{1}{2}$, apply the transformation T: $\tau^*=\tau^*+1$ until this condition is first violated. Go to 1.

5. If $|\tau^*|<1$ apply the transformation $S:\tau^*=-1/\tau^*$. Go to 1.

6. If $\mathrm{Re}\,\tau^*>\frac{1}{2}$, apply the transformation T^{-1}: $\tau^*=\tau^*-1$ until this condition is first violated. Go to 1.

The computational procedure we have described still needs an effective algorithm to evaluate the function $J(\tau)$ and possibly also its derivative. To that end we will use yet another modular function $\lambda: H\to \mathbf{C}$ [2, 3]. Though not invariant under Γ, it is invariant under a subgroup $\Gamma_2\subset\Gamma$, also known as the λ-group [4].

It is important that, instead of calculating $g_2(1,\tau)$, $g_3(1,\tau)$ for each $\tau^*\in H$, and only then $J(\tau)$ we can use the following formula [2, 3]

$$J(\tau)=\frac{4(\lambda^2(\tau)-\lambda(\tau)+1)^3}{27\lambda^2(\tau)(1-\lambda(\tau))^2} \tag{3.5}$$

$$\lambda(\tau)=1-\prod\left[\frac{1-q^{2n-1}}{1-q^{2n+1}}\right]^8 \quad [q=e^{i\pi\tau}] \tag{3.6}$$

Here, and throughout what follows, the product ranges over $n=1, 2, \ldots$. The second formula enables us to calculate $\lambda(\tau)$ fairly reliably for $\tau\in H$—especially if the initial approximation is chosen sufficiently distant from the axis $\mathrm{Im}\,\tau=0$.

The value of the parameter q may be determined by other methods, applying several iterative procedures [5] to determine the roots of the equation

$$4z^3-g_2z-g_3=0 \tag{3.7}$$

and to calculate the complete elliptic integrals of the first kind.

Once the value of $\tau\in G$ satisfying Eq. (3.4) has been determined, we must find the half-period ω of least absolute value. We recall that the problem has already been solved for $\tau=\infty$, in which case the elliptic function degenerates into a singly periodic function: $\omega'=\infty$. We may therefore assume [4] that the roots of Eq. (3.4) have the form

$$e_1=2a, \quad e_2=e_3=-a$$

where necessarily

$$g_2=12a^2, \quad g_3=8a^3, \quad \omega=(12a)^{-1/2}\pi$$

If $J(\tau)=0$, we have $\tau=e^{2\pi i/3}$. Then $g_2(\omega, \omega')=0$ and, by formulae (3.1), ω may be determined using the function $g_3(\omega, \omega')$. We have

$$b=\omega^{-6}g_3(1,\tau) \tag{3.8}$$

The quantity b is given, while $g_3(1,\tau)$ may be calculated in terms of theta-functions [4]

$$g_3(1,\tau)=\frac{4}{27}\left[\frac{\pi}{2}\right]^6[\theta_2^4(0|\tau)+\theta_3^4(0|\tau)][\theta_3^4(0|\tau)+\theta_0^4(0|\tau)][\theta_0^4(0|\tau)-\theta_2^4(0|\tau)] \tag{3.9}$$

where the theta-functions are calculated by the convenient formulae [2, 4]

$$\begin{aligned}
&\theta_1(\upsilon|\tau)=2q_0q^{1/4}\sin\pi\upsilon\prod Q_{2n}^-\\
&\theta_2(\upsilon|\tau)=2q_0q^{1/4}\cos\pi\upsilon\prod Q_{2n}^+\\
&\theta_3(\upsilon|\tau)=q_0\prod Q_{2n-1}^+, \quad \theta_0(\upsilon|\tau)=q_0\prod Q_{2n-1}^-\\
&q_0=\prod[1-q^{2n}], \quad Q_m^\pm=1\pm 2q^m\cos 2\pi\upsilon+q^{2m}, \quad q=e^{i\pi\tau}
\end{aligned} \tag{3.10}$$

The half-period ω may be any one of the six roots of Eq. (3.8), the corresponding numbers being

$$\omega = \left|\frac{g_3(1,\tau)}{b}\right| \exp\left[\frac{i}{6}\arg\left(\left|\frac{g_3(1,\tau)}{b}\right|\right)\right], \quad \omega' = \tau\omega = e^{2i\pi/3}\omega, \quad \omega+\omega'$$

These points, together with the three points centrally symmetric to them, form a complete set of sixth roots of $g_3(1, \tau)/b$. Any of these six numbers is a suitable half-period ω. The second half-period will then be equal to $\tau\omega$, and the doubly periodic lattice will not be changed by a different choice.

If $J(\tau)=1$, then $\tau=i$ is a solution. This is possible if $g_3(w, \omega')=0$. Then we can again determine ω from (3.1), using the equation

$$a = \omega^{-4} g_2(1, \tau) \tag{3.11}$$

in which we must put [4]

$$g_2(1,\tau) = 2/3(\pi/2)^4[\theta_2^8(0|\tau)+\theta_3^8(0|\tau)+\theta_0^8(0|\tau)]$$

The half-period here may be any of the four roots of Eq. (3.11), which lie at the points

$$\omega = \left|\frac{g_2(1,\tau)}{a}\right| \exp\left[\frac{i}{4}\arg\left(\left|\frac{g_2(1,\tau)}{a}\right|\right)\right], \quad \omega' = \tau\omega = i\omega, \quad -\omega, \quad i\omega$$

Taking any of these numbers as ω, we must automatically take ω' to be $i\omega$. Hence the integral lattice will not depend on the specific choice made.

In all other situations, i.e. when a, $b \neq 0$, combining (3.8) and (3.11), we obtain

$$\omega^2 = (\beta/\alpha) \quad (\alpha = g_2(1, \tau)/a., \quad \beta = g_3(1, \tau)/b)$$

As half-period we can choose one of the two numbers

$$\omega = |\beta/\alpha| \exp[\tfrac{i}{2}\arg(\beta/\alpha)], \quad -\omega$$

Then the second half-period will be, accordingly, either $\tau\omega$ or $-\tau\omega$. The integral lattice obtained as a result does not depend on the specific choice made. In addition, ω and $-\omega$. satisfy both Eqs (3.8) and (3.11), since

$$\omega^4 = \beta^2/\alpha^2 = \alpha^3/\alpha^2 = \alpha, \qquad \omega^6 = \beta^3/\alpha^3 = \beta^3/\beta^2 = \beta$$

where, in accordance with (3.8) and (3.11), allowance is made for the fact that $\beta^2=\alpha^3$. The other roots of Eqs (3.8) and (3.11) cannot be identical. This is not surprising, since the roots of the former equation form a square, while those of the latter are arranged in a regular hexagon. The vertices may therefore coincide at only two points.

It remains to see how to calculate the ℘-functions and their derivatives in formulae (2.11) or (2.21). We again use theta-functions [4], obtaining

$$\wp[u_1] = e_\alpha + \frac{1}{4\omega^2}\left(\frac{\theta_1'(0|\tau)}{\theta_{\alpha+1}(0|\tau)}\frac{\theta_{\alpha+1}(\upsilon|\tau)}{\theta_1(\upsilon|\tau)}\right)^2 \quad (\alpha = 1,2,3)$$

$$\wp'[u_1] = -\frac{1}{4\omega^3}[\theta_2(0|\tau)\theta_3(0|\tau)\theta_0(0|\tau)]^2 \frac{\theta_2(\upsilon|\tau)\theta_3(\upsilon|\tau)\theta_0(\upsilon|\tau)}{\theta_1^3(\upsilon|\tau)}$$

$$\omega_1 = \omega, \quad \omega_2 = \omega', \quad \omega_3 = \omega+\omega', \quad \theta_4(\upsilon|\tau) \equiv \theta_0(\upsilon|\tau), \quad \upsilon = u_1/(2\omega)$$

where $e_\alpha = \wp[\omega_\alpha]$ are the roots of Eq. (3.7)

Thus, to calculate the $\wp$-function we have to know at least one root of Eq. (3.7). The roots may also be expressed in terms of theta-functions [4]

$$e_1 = \frac{\pi^2}{12\omega^2}[\theta_3^4(0|\tau)+\theta_0^4(0|\tau)]$$

$$e_2 = \frac{\pi^2}{12\omega^2}[\theta_2^4(0|\tau)-\theta_0^4(0|\tau)]$$

$$e_3 = -e_1 - e_2 = -\frac{\pi^2}{12\omega^2}[\theta_2^4(0|\tau)+\theta_3^4(0|\tau)]$$

After determining τ and ω in the general position when $\alpha_1 \neq 0$, we still have to determine the parameter characterizing the uniformization of the appropriate elliptic curve: the solution of the system of equations (2.10). We note, finally, that ω is the half-period of the $\wp$-function in formulae (2.11) when $\alpha_1 \neq 0$. By formula (2.11), a doubly periodic solution of Eq. (2.1) has half-period $\alpha_1^{-1/2}\omega$. Thus the three numbers τ, $\alpha_1^{-1/2}\omega$, υ are structural constants that uniquely define a doubly periodic solution of Eq. (2.1).

4. COMPUTING THE TRAJECTORIES

To complete the description of the solution of the problem of two fixed centres, we need a procedure for computing the functions $t(u)$ (in Eq. (1.2)) and $\nu(u)$ (in Eq. (1.4)).

It is obvious from (1.2) and (1.4) that $\nu(u)$ and $t(u)$ may be evaluated with the help of the elliptic integrals

$$\int \frac{du}{z_2^2(u)-\gamma^2},\quad \int z_2^2(u)du \quad (\gamma = 1, i) \tag{4.1}$$

where the function $z_2(u)$ is of the form $z_2(u) = z(\upsilon_0 + u)$, obtained from (2.12), and υ_0 is obtained from (2.13).

The first integral of (4.1) can be reduced to computing the integrals

$$I = \int \frac{du}{z_2(u)-\gamma'} \quad (\gamma' = \pm 1, \pm i) \tag{4.2}$$

Changing from u to the new variable of integration

$$u_1 = \alpha_1^{1/2}(\upsilon_0 + u) \tag{4.3}$$

we can reduce the integral (4.2) to

$$I_1 = \int \frac{du_1}{z_1(u_1)-c}$$

where $z_1(u_1)$ is obtained from (2.11).

It follows from the standard identity [2]

$$\frac{1}{2}\frac{\wp'(u-\upsilon/2)-\wp'(\upsilon)}{\wp(u-\upsilon/2)-\wp(\upsilon)} = \frac{1}{2}\frac{\wp'(u-\upsilon/2)+\wp'(u+\upsilon/2)}{\wp(u-\upsilon/2)-\wp(u+\upsilon/2)}$$

that $z_i(u_1)$ is an even function and $z_1'(u_1)$ an odd function. The function $z_1(u_1)$ has two first-order poles in the parallelogram of periods: $a_1 = \upsilon/2$, $a_2 = -\upsilon/2$. The same is true of the zeros of the function $z_1(u_1) - c$, hence also of the poles of the integrand $(z_1(u_1) - c)^{-1}$. Suppose these are the points $b_1 = w$, $b_2 = -w$.

Taking the properties of elliptic functions into account, we finally obtain

$$\frac{1}{z_1(u_1) - c} = \frac{\zeta(b_1) - \zeta(b_2)}{\wp(b_1) - \wp(b_2)} + \frac{\zeta(u_1 - b_1) - \zeta(u_1 - b_2)}{\wp(b_1) - \wp(b_2)}$$

$$b_1 = w - \upsilon/2, \quad b_2 = -w - \upsilon/2$$

where $\zeta(u_1)$ is the familiar Weierstrass ζ-function. Finally, evaluation of the integral I_1 yields

$$I_1(u_1) = \left(C_0 - 2C_1 \frac{\eta w}{\omega}\right)(u_1 - \upsilon_{10}) +$$
$$+C_1\left[\ln\theta_1\left(\frac{u_1 - b_1}{2\omega}\right) - \ln\theta_1\left(\frac{u_1 - b_2}{2\omega}\right) + \ln\theta_1\left(\frac{\upsilon_{10} - b_2}{2\omega}\right) - \ln\theta_1\left(\frac{\upsilon_{10} - b_1}{2\omega}\right)\right]$$
$$C_0 = \frac{\zeta(w - \upsilon/2) + \zeta(w + \upsilon/2)}{\wp(w - \upsilon/2) - \wp(w + \upsilon/2)}, \quad C_1 = \frac{1}{z_1'(w)}$$

For the ζ-function we propose to use the equality

$$\zeta(u|\tau, \omega) = \frac{\eta}{\omega}u + \frac{1}{2\omega}\frac{\theta_1'(u/2\omega)}{\theta_1(u/2\omega)}$$

The constant γ may also be determined by the useful formula [2]

$$\eta = \frac{\pi^2}{12\omega}\left[1 - 24\sum_{n=1}^{\infty}\frac{q^{2n}}{(1 - q^{2n})^2}\right] \quad [q = e^{i\pi\tau}]$$

Now consider the quadrature for the time variable. This reduces to evaluating the second integral in (4.1). If we now make the substitution (4.3), the integrand may be expressed in terms of (2.12). Taking the properties of the poles $z_1(u_1)$ into account, we finally obtain

$$\left(z_1(u_2) - \frac{\alpha}{4\alpha_1}\right)^2 = A_0 + A_1(\zeta(u_1 - \upsilon/2) - \zeta(u_1 + \upsilon/2)) +$$
$$+A_2\wp(u_1 - \upsilon/2) + A_3\wp(u_1 + \upsilon/2) \tag{4.4}$$

and the constants A_k $(k = 0, 1, 2, 3)$ may be determined in the form

$$A_0 = \frac{\alpha^2}{16\alpha_1^2} + \zeta(\upsilon)\frac{\alpha}{2\alpha_1} - \wp(\upsilon), \quad A_1 = \frac{\alpha}{2\alpha_1}, \quad A_2 = A_3 = 1$$

The function (4.4) is integrated in standard fashion.

We draw attention to a well-known fact: the functions $\nu(u)$ and $t(u)$ consist of two terms, one linear and one periodic as functions of u. this may be utilized when one wishes to invert the function $t(u)$ to calculate the coordinates λ, μ, ν for a given instant of time.

The equation

$$t(u) = t - t_0 \tag{4.5}$$

may be solved as follows. Let

$$t(u) = lu + \phi(u)$$

where $\phi(u)$ is periodic in u and l is a constant. Then, knowing the increment of $t(u)$ over each of the half-periods 2ω and $2\tau\omega$ (with respect to the variable u, not u_1, as before), namely, $2\omega l$, $2\tau\omega l$, respectively, we can solve Eq. (4.5) in the fundamental parallelogram of periods. Suppose this equation has the form

$$t(u) = t_1 \qquad (t_1 = t - t_0 - 2n\omega l - 2n'\tau\omega l) \tag{4.6}$$

To solve this equation, one can take the initial approximation to be $u_0 = l^{-1}t_1$, and then employ an iterative procedure to determine a solution. If this solution is found to be u_*, the final answer will have the form $u = u_* + 2n\omega + 2n'\tau\omega$. The iterative computation of u_* is based on the non-degeneracy property of the derivative dt/du. Referring to (1.2), we conclude that degeneracy can occur in real motions at just one point—the origin. For the space-satellite case, therefore, Eq. (4.6) is always solvable in a rigorous sense.

As to an algorithm to compute the number $t_1 \in \mathbf{C}$, since the periods 2ω, $2\omega' = 2\tau\omega$ form the basis of a real vector space in the complex plane, it follows that for $u \in \mathbf{C}$ we have a representation $u = a2\omega + a'2\omega'$ $(a,\ a' \in \mathbf{R})$. The vectors 2ω and $2\omega'$ are taken by the linear mapping $u \to lu$ to the vectors $2l\omega$ and $2l\omega'$, respectively. This transformation preserves linear independence. We therefore obtain a unique representation

$$t - t_0 = b2l\omega + b'2l\omega' \quad (b,\ b' \in \mathbf{R})$$

Clearly, to transform to the fundamental parallelogram, we must replace b and b' by their fractional parts: their integer parts will be precisely the numbers n and n' in (4.6): $n = [b]$, $n' = [b']$.

It remains to find $b,\ b' \in \mathbf{R}$. Let

$$t - t_0 = t_1 + t_2 i, \qquad l\omega = \Omega_1 + \Omega_2 i, \qquad l\omega' = \Omega_1' + \Omega_2' i$$

Then b, b' is a solution of the system of linear equations

$$2\Omega_1 b + 2\Omega_1' b' = t_1, \quad 2\Omega_2 b + 2\Omega_2' b' = t_2$$

which is uniquely solvable, since the vectors $l\omega$ and $l\omega'$ are linearly independent in $\mathbf{C} \simeq \mathbf{R}^2$.

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